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Using Fractional Series for Solving Fractional Burger's Equation

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Abstract

In this paper we study Fractional Burger's Equation of the form $u_t + uu_x - u_x^{(\alpha)}$=0, $x\in\mathbb{R}$, t>0 and solving it by method namely fractional power series. Statistical concepts are used to show that our solution agrees with nature.

Keywords: Fractional calculus, Fractional power series, Burger's equation, Traveling wave solutions, Probability density function.

1. Introduction

It is well known that a large number of mathematical models were classified by differential equations (partial or ordinary) of integer or fractional order with initial or boundary or initial and boundary conditions especially applications in the areas of physics and engineering, chemistry,…, etc. such as electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional partial differential equations. In general , there exists no method that yields an exact Solution for a fractional partial differential equation . Since most of the nonlinear fractional partial differential equations cannot be solved exactly, thus approximate and numerical methods must be used.

Burger's equations are used and study for Control of flows and It has a large variety of applications in modeling of water in unsaturated oil, dynamics of soil in water, statics of flow problems, mixing and turbulent diffusion, cosmology and seismology. The one dimensional nonlinear Burger's equation was first introduced by (Bateman 1915), who found its steady solutions descriptive of certain viscous flows. It was later proposed by (Burger1948) , as one of a class of equation describing mathematical models of turbulence. In the context of gas dynamics it was discussed by(Hopf , and Cole1950) . In recent years many researchers have used various numerical methods specially based on finite difference, finite element boundary element techniques and direct variational method to solve Burger ′s equation . (E. Benton and platzman) surveyed exact solution of one dimensional Burger ′s equation. In (1997 D.S. Zhang , G.W. Wei and D. J. Kouri) , solved it for high Reynolds number , this simple approach can provide very high accuracy while using a small number of grid points . In (2005A. Gorguis) gives comparison between Cole - Hopf transformation and Decomposition method for solving Burger ′s equation. In(2006 Jerome I. V. Lewandowski) used Marker method which relies on the definition of Convective field associated with the underlying PDE, the information About the approximate solution is associated with the response of convective field. In (2006K. Altiparmak) gave Economized Rational approximation method using pade ′s approximation which is efficient than Rational approximation. In (2006 M. K. Kadalbajoo and A. Awasti) developed stable numerical method based on Crank Nicolson to solve Burger's equation. In(2009 J. Biazar and H. Aminikhah) solve Burger's equation by using variational iteration method by which Approximate solution can be found and which is better than ADM. In (2008 J. K. Djoko) examine the stability of a finite difference Approximation for Burger's equation by approximating the nonlinear term by a linear expression using techniques based on the boundaries of the solution sequence with respect to Δt for $t\in$ (0 , ∞) and with the help of discrete Aronwall lemma stability is achieved. In (2009 Sachin S.Wani and Sarita Thakar) analysed stability of Mixed Euler Method for one Dimensional nonlinear Burger's equation. In (2009 K. Pandey and L Verma and A . K. Verma) wrote on difference scheme for Burger's equation. In (2011 Kanti Pandey and Lajja Verma) gave a note on Crank Nicolson scheme for Burger's equation without Hopf -Cole transformation solutions are obtained by ignoring nonlinear term. In (2014[8], Saad N. and Muna Saleh) used Bernoulli equation to solve Burger's equation .

2. Preliminaries

In this section, we provide some basic definitions and properties of the fractional calculus theory which are used in this paper.

Definition 2.1[2]

A real function f(x), x>0, is said to be in the space C_μ, $\mu\in\mathbb{R}$, if there exists a real number p, p>μ, such that f(x)=$x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x)\in C_\mu$, n$\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Definition 2.2 [2]

The Riemann-Liouville fractional integral operator of order α≥0,of a function f(x)∈C_μ, $\mu\geq-1$is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^1(x-t)^{\alpha-1}f(t)dt,\ x>0 \tag{1}$$

$$J^0 f(x)=f(x).$$

Properties of the operator J^αcan be found for f∈C_μ, $\mu\geq-1$, α, $\beta\geq0$ and $\gamma>-1$, we have

1. $J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x)=J^\beta J^\alpha f(x)$

2. $J^\alpha C=\frac{C}{\Gamma(\alpha+1)}x^\alpha$

3. $J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$

Definition 2.3 [5]

The Caputo definition of fractional derivative operator is given by

$$D^\alpha f(x)=J^{n-\alpha}D^n f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}f^{(n)}(t)dt\ ,\alpha>0 \tag{2}$$

For $n-1<\alpha\leq n$, $n\in\mathbb{N}$, x>0and Γ(.) is the gamma function.

Definition 2.4[5]

For n be the smallest integer that exceeds α, the Caputo fractional derivative of a function$u$ $(x,)$ of order α>0 is defined by

$$D^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^x(x-\tau)\frac{\partial^n u(x,\tau)}{\partial\tau^n}d\tau, n-1<\alpha<n\\ \frac{\partial^n u(x,t)}{\partial x^n},\ \ \alpha=n\in\mathbb{N}\end{cases} \tag{3}$$

and satisfies the following properties:

1. $D^\alpha C=0$, C constant

2. $D^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}$, x>0, $\gamma>-1$

3. $D^\alpha(\sum_{i=0}^m c_i f_i(x,t))=\sum_{i=0}^m c_i D^\alpha f_i(x,t)$,where $c_0,c_1,\ldots,c_m$ are constant.

Lemma 2.1[5]

If $n-1<\alpha\leq n$, f∈C_μ^n, $n\in\mathbb{N}$ and $\mu\geq-1$, then

$D^\alpha J^\alpha f(x)=f(x)$ and

$J^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{n-1}f^k(0^+)\frac{x^k}{k!}$, where x>0

3. Fractional Power Series

In this section, we will use the fractional power series of the form:

$$u(x,t)=\sum_{m=0}^\infty\sum_{n=0}^\infty A_{mn}x^{m\alpha}t^n \tag{4}$$

To solve the fractional Burger equation of the form:

$$u_x^{(\alpha)} = uu_x + u_t \qquad 0 < \alpha \leq 1, \quad x \in \mathbb{R}, \quad t \geq 0 \tag{5}$$

Differentiate (4) to get $u_x^{(\alpha)}$, u_x and u_t and substituting them in (5)

Equating the corresponding coefficients to get the following system:

$$A_{00}\, A_{10} = 0 \tag{6}$$

$$2A_{00}\, A_{20} + A_{10}^2 = 0 \tag{7}$$

$$3A_{00}\, A_{30} + 3A_{10}\, A_{20} = 0 \tag{8}$$

$$4A_{00}\, A_{40} + 4A_{10}\, A_{30} + 2A_{20}^2 = 0 \tag{9}$$

$$5A_{00}\, A_{50} + 5A_{10}\, A_{40} + 5A_{20}\, A_{30} = 0 \tag{10}$$

$$6A_{00}\, A_{60} + 6A_{10}\, A_{50} + 6A_{20}\, A_{40} + 3A_{30}^2 = 0 \tag{11}$$

$$7A_{00}\, A_{70} + 7A_{10}\, A_{60} + 7A_{20}\, A_{50} + 7A_{30}\, A_{40} = 0 \tag{12}$$

$$8A_{00}\, A_{80} + 8A_{10}\, A_{70} + 8A_{20}\, A_{60} + 8A_{30}\, A_{50} + 4A_{40}^2 = 0 \tag{13}$$

From the first equation of (6) $A_{00} = 0$ or $A_{10} = 0$

Or $A_{00} = A_{10} = 0$ then we get the trivial solution

Therefore we consider (7)

$2A_{00}\, A_{20} + A_{10}^2 = 0$

So $\quad A_{20} = -\frac{A_{10}^2}{2A_{00}}$

Similarly $A_{30} = \frac{A_{10}^3}{2A_{00}^2}$, $A_{40} = -\frac{5A_{10}^4}{8A_{00}^3}$ $A_{50} = \frac{7A_{10}^5}{8A_{00}^4}$, $A_{60} = -\frac{21A_{10}^6}{16A_{00}^5}$, $A_{70} = \frac{33A_{10}^7}{16A_{00}^6}$, $A_{80} = -\frac{429A_{10}^8}{128A_{00}^7}$, and so on.

After substituting these coefficients into (4) then the solution of (5) is

$$\begin{aligned}
u(x, t) = {} & A_{00} + A_{10}\, x^{\alpha} - \frac{A_{10}^2}{2A_{00}} x^{2\alpha} + \frac{A_{10}^3}{2A_{00}^2} x^{3\alpha} - \frac{5A_{10}^4}{8A_{00}^3} x^{4\alpha} + \frac{7A_{10}^5}{8A_{00}^4} x^{5\alpha} - \frac{21A_{10}^6}{16A_{00}^5} x^{6\alpha} \\
& + \frac{33A_{10}^7}{16A_{00}^6} x^{7\alpha} - \frac{429A_{10}^8}{128A_{00}^7} x^{8\alpha} + \ldots + \Gamma(\alpha+1)A_{10}t - \frac{\Gamma(2\alpha+1)\, A_{10}^2}{4A_{00}} t^2 \\
& + \frac{\Gamma(3\alpha+1)\, A_{10}^3}{12A_{00}^2} t^3 - \frac{5\Gamma(4\alpha+1)\, A_{10}^4}{192A_{00}^3} t^4 + \frac{7\Gamma(5\alpha+1)\, A_{10}^5}{960A_{00}^4} t^5 - \frac{21\Gamma(6\alpha+1)\, A_{10}^6}{11520A_{00}^5} t^6 \\
& + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{80640A_{00}^6} t^7 - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{5160960A_{00}^7} t^8 + \ldots - \frac{\Gamma(2\alpha+1)\, A_{10}^2}{2\Gamma(\alpha+1)A_{00}} x^{\alpha} t + \frac{\Gamma(3\alpha+1)\, A_{10}^3}{4\Gamma(\alpha+1)A_{00}^2} x^{\alpha} t^2 \\
& - \frac{5\Gamma(4\alpha+1)\, A_{10}^4}{48\Gamma(\alpha+1)A_{00}^3} x^{\alpha} t^3 + \frac{7\Gamma(5\alpha+1)\, A_{10}^5}{192\Gamma(\alpha+1)A_{00}^4} x^{\alpha} t^4 - \frac{21\Gamma(6\alpha+1)\, A_{10}^6}{1920\Gamma(\alpha+1)A_{00}^5} x^{\alpha} t^5 \\
& + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{11520\Gamma(\alpha+1)A_{00}^6} x^{\alpha} t^6 - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{645120\Gamma(\alpha+1)A_{00}^7} x^{\alpha} t^7 + \ldots + \frac{\Gamma(3\alpha+1)\, A_{10}^3}{2\Gamma(2\alpha+1)A_{00}^2} x^{2\alpha} t \\
& - \frac{5\Gamma(4\alpha+1)\, A_{10}^4}{16\Gamma(2\alpha+1)A_{00}^3} x^{2\alpha} t^2 + \frac{7\Gamma(5\alpha+1)\, A_{10}^5}{48\Gamma(2\alpha+1)A_{00}^4} x^{2\alpha} t^3 - \frac{21\Gamma(6\alpha+1)\, A_{10}^6}{384\Gamma(2\alpha+1)A_{00}^5} x^{2\alpha} t^4 \\
& + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{1920\Gamma(2\alpha+1)A_{00}^6} x^{2\alpha} t^5 - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{92160\Gamma(2\alpha+1)A_{00}^7} x^{2\alpha} t^6 + \ldots - \frac{5\Gamma(4\alpha+1)\, A_{10}^4}{8\Gamma(3\alpha+1)A_{00}^3} x^{3\alpha} t \\
& + \frac{7\Gamma(5\alpha+1)\, A_{10}^5}{16\Gamma(3\alpha+1)A_{00}^4} x^{3\alpha} t^2 - \frac{21\Gamma(6\alpha+1)\, A_{10}^6}{96\Gamma(3\alpha+1)A_{00}^5} x^{3\alpha} t^3 + \frac{33\Gamma(7\alpha+1)A_{10}^7}{384\Gamma(3\alpha+1)A_{00}^6} x^{3\alpha} t^4 \\
& - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{15360\Gamma(3\alpha+1)A_{00}^7} x^{3\alpha} t^5 + \ldots + \frac{7\Gamma(5\alpha+1)\, A_{10}^5}{8\Gamma(4\alpha+1)A_{00}^4} x^{4\alpha} t - \frac{21\Gamma(6\alpha+1)\, A_{10}^6}{32\Gamma(4\alpha+1)A_{00}^5} x^{4\alpha} t^2 \\
& + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{96\Gamma(4\alpha+1)A_{00}^6} x^{4\alpha} t^3 - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{3072\Gamma(4\alpha+1)A_{00}^7} x^{4\alpha} t^4 + \ldots + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{32\Gamma(5\alpha+1)A_{00}^6} x^{5\alpha} t^2 \\
& - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{768\Gamma(5\alpha+1)A_{00}^7} x^{5\alpha} t^3 + \ldots + \frac{33\Gamma(7\alpha+1)\, A_{10}^7}{16\Gamma(6\alpha+1)A_{00}^6} x^{6\alpha} t - \frac{429\Gamma(8\alpha+1)\, A_{10}^8}{256\Gamma(6\alpha+1)A_{00}^7} x^{6\alpha} t^2 + \ldots
\end{aligned} \tag{14}$$

$- \frac{429\Gamma(8\alpha+1)\, A_{10}^{8}}{128\Gamma(7\alpha+1) A_{00}^{7}} x^{7\alpha} t \ + \ldots$

4. Statistical Tests for Reliability of the Solution

Let the traveling wave solution (14) be a probability density function if:

$$\int_0^1 \int_0^1 u(x\,,t)dxdt = 1 \qquad (15)$$

Because of the uniform convergence few terms are enough for good accuracy [9]. So we consider

$$\int_0^1 \int_0^1 [A_{00} + A_{10}x^{\alpha} - \frac{A_{10}^2}{2A_{00}} x^{2\alpha} + \frac{A_{10}^3}{2A_{00}^2} x^{3\alpha} - \frac{5A_{10}^4}{8A_{00}^3} x^{4\alpha} + \frac{7A_{10}^5}{8A_{00}^4} x^{5\alpha} - \frac{21A_{10}^6}{16A_{00}^5} x^{6\alpha}]dxdt = 1$$

That is

$$A_{00} + \frac{A_{10}}{(\alpha+1)} - \frac{A_{10}^2}{2(2\alpha+1)} + \frac{A_{10}^3}{2(3\alpha+1)A_{00}^2} - \frac{5A_{10}^4}{8(4\alpha+1)A_{00}^3} + \frac{7A_{10}^5}{8(5\alpha+1)A_{00}^4} - \frac{21A_{10}^6}{16(6\alpha+1)A_{00}^5} = 1 \qquad (16)$$

In which there are three degrees of freedom A_{00}, A_{10} and α.

Let A_{00}= 1and $\alpha = \frac{3}{4}$ then one of the values of A_{10} is zero and the solution is $u(x\,,t) = 1$ but we want non constant solution ,therefor by using numerical analysis such as (Newton –Raphson method[4]) to find another value of A_{10} from (16) which has one solution

A_{10} = 1.2389

Substituting these values into equation (14) then the probability density function is:

$$u(x\,,t) = 1 + 1.2389\,x^{3/4} - 0.7674\,x^{3/2} + 0.9508\,x^{9/4} - 1.4724\,x^{3}$$
$$+ 2.5538\,x^{15/4} - 4.7459\,x^{9/2} + 1.1385t - 0.5099t^2 + 0.4039t^3$$
$$- 0.3681\,t^4 + 0.3529t^5 - 0.3450t^6 + 0.3389t^7 - 0.3322t^8$$
$$- 1.1098\,x^{3/4}t + 1.3186x^{3/4}t^2 - 1.3531x^{3/4}t^3 + 1.9205x^{3/4}t^4$$
$$- 2.2526x^{3/4}t^5 + 2.1801x^{3/4}t^6 + 3.2209x^{3/2}t - 5.8704x^{3/2}t^2$$
$$+ 9.3822x^{3/2}t^3 - 13.7559x^{3/2}t^4$$

(I) the moments

To evaluate the expected values E(x), E(t), E(xt) and the second moments E(x^2), E(t^2) we need $u^*(x)$ and u^{**}(t) .we find them as follows:

$$u^*(x) = \int_0^1 u(x\,,t)dt$$
$$= 1 + 1.2389\,x^{3/4} - 0.7674\,x^{3/2} + 0.9508\,x^{9/4} - 1.4724\,x^3$$
$$+ 2.5538\,x^{15/4} - 4.7459\,x^{9/2} + 0.5693 - 0.1699 + 0.1009 - 0.0736$$
$$+ 0.0588 - 0.0493 + 0.0424 - 0.0369 - 0.5549\,x^{3/4} + 0.4395\,x^{3/4}$$
$$- 0.3383\,x^{3/4} + 0.3841\,x^{3/4} - 0.3754\,x^{3/4} + 0.3114\,x^{3/4}$$
$$+ 1.6105\,x^{3/2} - 1.9568\,x^{3/2} + 2.3456x^{3/2} - 2.7512x^{3/2}$$

$$u^{**}(t) = \int_0^1 u(x\,,t)dx$$
$$= 1 + 0.7079 - 0.3069 + 0.2926 - 0.3681 + 0.5376 - 0.8629 + 1.1385t$$

$-0.5099t^{2}+0.4039t^{3}-0.3681t^{4}+0.3529t^{5}-0.3450t^{6}$

$+0.3389t^{7}+0.3322t^{8}-0.6342t+0.7535t^{2}-0.7732t^{3}+1.0974t^{4}$

$-1.2872t^{5}+1.2458t^{6}+1.2884t-2.3482t^{2}+3.7529t^{3}-5.5024t^{4}$

(1) Expected Value of x

$$E(x) = \int_0^1 x u^*(x)\,dx = 0.2418$$

(2) Expected Value of t

$$E(t) = \int_0^1 t u^{**}(t)\,dt = 0.4361$$

It means that the first expected length of the wave is concentrated at value (power point) 0.2418, which is the middle of the wave which agrees with nature.

Also the first expected time of wave is concentrated at 0.4361, which means that the wave takes long time which agree with nature.

(3) The second moment of x

$$E(x^2) = \int_0^1 x^2 u^*(x)dx = 0.1190$$

(4) The second moment of t

$$E(t^2) = \int_0^1 t^2 u^{**}(t)dt = 0.2296$$

$E(x)>E(x^2)$ which shows that the first wave is stronger than the second

The second expected length of the wave is concentrated at the power point 0.1190, which means that the length begins to disperse (scatter).

The second expected time of the wave is concentrated at 0.2296, which means that the wave stay for short time which agree with nature.

(5) The Expected Value of xt

$$E(xt) = \int_0^1\int_0^1 xt\, u(x,t)dxdt = 0.0831$$

This joint expected value for length and time of the wave is 0.0831, which means moderate which agree with nature.

(II) The Variance

(1) Variance of x

$$\sigma_x^2 = E(x^2) - [E(x)]^2 = 0.0394$$

(2) Variance of t

$$\sigma_t^2 = E(t^2) - [E(t)]^2 = 0.0605$$

The variation for length of wave is 0.0605, so that the separation is very small. This means that the power of wave is focused in the middle of the wave and separated begins from first wave which agrees with nature.

The variation for time of wave is 0.0394, so that the separation is very small. This means that the time of separated wave begins from first wave and so on which agree with nature.

(III) The Covariance

$$Cov(x,t) = E(xt) - E(x)E(t) = -0.0223$$

The range of deviation of the length and time of the wave from its expected values is very small which agrees with nature.

(IV) The Correlation Coefficients

$$\rho = \frac{cov(x,t)}{\sqrt{var(x)}\sqrt{var(t)}}$$

$= -0.4569$

This means that the relation between the amplitude of the wave and time is strong (high amplitude corresponding to the beginning of the wave in terms of length and time) and vice versa which agree with nature.

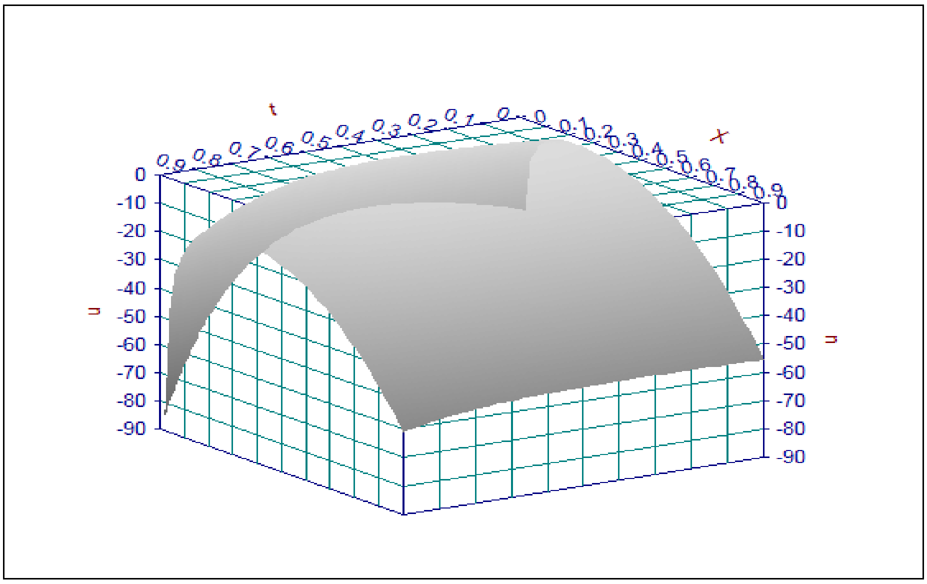

Figure 1. represents the traveling wave solution when $0<x<1$, $0<t<1$ and $-90<u<0$.

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